

On qualitative robustness of the Lotka–Nagaev estimator for the offspring mean of a supercritical Galton–Watson process

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Abstract

We characterize the sets of offspring laws on which the Lotka–Nagaev estimator for the mean of a supercritical Galton–Watson process is qualitatively robust. These are exactly the locally uniformly integrating sets of offspring laws, which may be quite large. If the corresponding global property is assumed instead, we obtain uniform robustness as well. We illustrate both results with a number of concrete examples. As a by-product of the proof we obtain that the Lotka–Nagaev estimator is [locally] uniformly weakly consistent on the respective sets of offspring laws, conditionally on non-extinction.

Keywords: Galton–Watson process, offspring mean, Lotka–Nagaev estimator, qualitative robustness, uniform conditional weak consistency, Strassen’s theorem, ψ -weak topology

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1 Introduction

A Galton–Watson branching process $(Z_n) := (Z_n)_{n \in \mathbb{N}_0}$ with initial state 1 and offspring distribution μ on $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ describes the evolution of the size of a population with initial size 1, where each individual i in generation k has a random number $X_{k,i}$ of descendants drawn from μ independently of all other individuals. In other words,

$$Z_0 := 1 \quad \text{and} \quad Z_n := \sum_{i=1}^{Z_{n-1}} X_{n-1,i} \quad \text{for } n \in \mathbb{N}. \quad (1)$$

For background see, for instance, [1, 2]. In this article we always assume that the mean

$$m_\mu := \sum_{k=1}^{\infty} k \mu[\{k\}]$$

of the offspring distribution μ is finite. A natural estimator for the offspring mean m_μ based on observations up to time n is the Lotka–Nagaev estimator [18, 21] given by

$$\hat{m}_n := \begin{cases} \frac{\sum_{i=1}^{Z_{n-1}} X_{n-1,i}}{Z_{n-1}} = \frac{Z_n}{Z_{n-1}} & , \quad Z_{n-1} > 0, \\ 0 & , \quad Z_{n-1} = 0. \end{cases} \quad (2)$$

This estimator requires knowledge only of the last two generation sizes Z_{n-1} and Z_n . Another popular estimator is the Harris estimator $\sum_{k=1}^n Z_k / \sum_{k=0}^{n-1} Z_k$, which is known to be the nonparametric maximum likelihood estimator for m_μ when observing all generation sizes Z_0, \dots, Z_n [9, 16] and even when observing the entire family tree [13]. However, in this article we restrict ourselves to the Lotka–Nagaev estimator. Note that from the point of view of applications it is often the case that the process cannot be observed for an extended period of time, such that the Lotka–Nagaev estimator is the simplest or indeed the only possible choice in these situations.

In the critical and subcritical cases, i.e. when $m_\mu \leq 1$, the mean cannot be estimated consistently due to the extinction of (Z_n) with probability 1. On the other hand, in the supercritical case, i.e. when $m_\mu > 1$, the Lotka–Nagaev estimator is strongly consistent on the set of non-extinction, which can be easily shown by adapting the argument of Heyde [14]. Asymptotic normality (assuming finite variance of the offspring law μ) on the set of non-extinction was obtained by Dion [6] among others. A discussion of further statistical properties can be found in [7]. For a recent overview of estimation in general branching processes we refer to [20].

The objective of the present article is to investigate the estimator \hat{m}_n for (qualitative) robustness in the supercritical case. Informally, the sequence (\hat{m}_n) is robust when a small change in μ results only in a small change of the law of the estimator \hat{m}_n uniformly in n . More precisely, given a set \mathcal{N} of probability measures μ on \mathbb{N}_0 with $m_\mu < \infty$, the

sequence of estimators (\widehat{m}_n) is said to be robust on \mathcal{N} if for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\mu_2 \in \mathcal{N}, \quad d(\mu_1, \mu_2) \leq \delta \quad \implies \quad \rho(\text{law}\{\widehat{m}_n|\mu_1\}, \text{law}\{\widehat{m}_n|\mu_2\}) \leq \varepsilon \quad \text{for all } n \in \mathbb{N}, \quad (3)$$

where d is any metric on \mathcal{N} which generates the weak topology and ρ is the Prohorov metric on the set \mathcal{M}_1^+ of all probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. The sequence (\widehat{m}_n) is said to be uniformly robust on \mathcal{N} if δ can be chosen independently of $\mu_1 \in \mathcal{N}$. [Uniform] robustness of (\widehat{m}_n) on \mathcal{N} means that the set of mappings $\{\mathcal{N} \rightarrow \mathcal{M}_1^+, \mu \mapsto \text{law}\{\widehat{m}_n|\mu\} : n \in \mathbb{N}\}$ is [uniformly] (d_{TV}, ρ) -equicontinuous. This definition is in line with Hampel's definition of robustness for empirical estimators in nonparametric statistical models [5, 11]. Note, however, that our situation is *not* covered by Hampel's setting, because our estimator \widehat{m}_n is not based on n i.i.d. observations. On the other hand, our setting is covered by the more general framework recently introduced in [24]. For background on robust statistics, see also [12, 15] and the references cited therein.

We point out that we do *not* claim that the Lotka–Nagaev estimator is particularly robust. For a “robustification” of the Lotka–Nagaev estimator, see [22]. We are rather interested in “how robust” the classical Lotka–Nagaev estimator is. To some extent, the degree of robustness of an estimator can be measured by the “size” of the sets \mathcal{N} on which the estimator is robust; see also [24]. Intuitively, the larger the sets \mathcal{N} on which the estimator is robust, the larger is the “degree” of robustness. Corollary 2.10 below gives an exact specification of these sets \mathcal{N} for the Lotka–Nagaev estimator. Similar investigations have recently been done by Cont et al. [4] (see also [17]) in the context of the empirical estimation of monetary risk measures. For instance, the empirical Value at Risk at level α (i.e., up to the sign, the empirical upper α -quantile) is robust on the set \mathcal{N} of all probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with a unique α -quantile; cf. Proposition 3.5 in [4].

Our main results state that the sets \mathcal{N} on which the sequence (\widehat{m}_n) is robust are exactly the locally uniformly integrating sets; and if a set \mathcal{N} is even uniformly integrating and satisfies $\inf_{\mu \in \mathcal{N}} m_\mu > 1$, then (\widehat{m}_n) is even uniformly robust on it. Uniformly integrating for a set \mathcal{N} means just that any set of random variables $\{Y \sim \mu : \mu \in \mathcal{N}\}$ is uniformly integrable. This property is just a tiny bit stronger than finiteness of $\sup_{\mu \in \mathcal{N}} m_\mu$; see Remark 2.3. Locally uniformly integrating means that every weakly convergent subsequence in \mathcal{N} is uniformly integrating.

In Section 2 we also provide various examples of (parametric) sets \mathcal{N} that are [locally] uniformly integrable. We illustrate the implied robustness statements in the context of estimating a parameter (via estimating the mean) that is either slightly perturbed or belongs to a model that is slightly misspecified. In both situations [uniform] robustness yields that the distribution of the estimator is largely unaffected.

2 Main results and discussion

For the exact formulation of our main results we have to define the Galton–Watson process as a sort of canonical process. More precisely, let $(Z_n) := (Z_n)_{n \in \mathbb{N}_0}$ be given by (1) with $(X_{k,i}) := (X_{k,i})_{(k,i) \in \mathbb{N}_0 \times \mathbb{N}}$ the coordinate process on

$$(\Omega, \mathcal{F}) := (\mathbb{N}_0^{\mathbb{N}_0 \times \mathbb{N}}, \mathfrak{P}(\mathbb{N}_0)^{\otimes (\mathbb{N}_0 \times \mathbb{N})})$$

(with \mathfrak{P} denoting the set of all subsets) under the product law

$$\mathbb{P}^\mu := \mu^{\otimes (\mathbb{N}_0 \times \mathbb{N})}.$$

Note that $(X_{k,i})$ is a double sequence of i.i.d. random variables with distribution μ .

Let \mathcal{N}_1^1 be the set of all probability measures μ on \mathbb{N}_0 with $m_\mu < \infty$, and d_{TV} the total variation distance on \mathcal{N}_1^1 , i.e.

$$d_{\text{TV}}(\mu_1, \mu_2) := \sup_{A \in \mathfrak{P}(\mathbb{N}_0)} |\mu_1(A) - \mu_2(A)| = \frac{1}{2} \sum_{k \in \mathbb{N}_0} |\mu_1[\{k\}] - \mu_2[\{k\}]|. \quad (4)$$

As before let \mathcal{M}_1^+ be the set of all probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and ρ be the Prohorov metric on \mathcal{M}_1^+ , i.e.

$$\rho(\mu_1, \mu_2) := \inf\{\varepsilon > 0 : \mu_1[A] \leq \mu_2[A^\varepsilon] + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R}_+)\} \quad (5)$$

with $A^\varepsilon := \{x \in \mathbb{R}_+ : \inf_{a \in A} |x - a| \leq \varepsilon\}$. Note that d_{TV} coincides with the Prohorov metric on \mathcal{N}_1^1 . In particular, d_{TV} and ρ metrize the weak topologies on \mathcal{N}_1^1 and \mathcal{M}_1^+ , respectively.

Definition 2.1 *For $\mathcal{N} \subset \mathcal{N}_1^1$, the sequence (\widehat{m}_n) is said to be robust on \mathcal{N} if for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \rho(\mathbb{P}^{\mu_1} \circ \widehat{m}_n^{-1}, \mathbb{P}^{\mu_2} \circ \widehat{m}_n^{-1}) \leq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

It is said to be uniformly robust on \mathcal{N} if δ can be chosen independently of $\mu_1 \in \mathcal{N}$.

Of course, the notion of robustness remains the same when replacing d_{TV} by any other metric metrizing the weak topology. The main result of this article is Theorem 2.4. For its formulation we need a version of Definition 3.3 in [24] concerning locally uniformly ψ -integrating sets. Here, we set $\psi(k) := k$, $k \in \mathbb{N}_0$. Note that choosing the identity function for ψ corresponds to the notion of locally uniformly integrating sets mentioned in the introduction. In our setting this choice is equivalent to considering ψ_1 when $\psi_p(k) := (1+k)^p$, $k \in \mathbb{N}_0$, $p \geq 0$ as introduced in (17) of [24]. This motivates the following definition and terminology.

Definition 2.2 A set $\mathcal{N} \subset \mathcal{N}_1^1$ is said to be locally uniformly ψ_1 -integrating if for every $\varepsilon > 0$ and $\mu_1 \in \mathcal{N}$ there exist some $\delta > 0$ and $\ell \in \mathbb{N}$ such that

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \sum_{k=\ell}^{\infty} k \mu_2[\{k\}] \leq \varepsilon.$$

It is said to be uniformly ψ_1 -integrating if for every $\varepsilon > 0$ there exists some $\ell \in \mathbb{N}$ such that

$$\sup_{\mu \in \mathcal{N}} \sum_{k=\ell}^{\infty} k \mu[\{k\}] \leq \varepsilon.$$

Remark 2.3 Any uniformly ψ_1 -integrating set \mathcal{N} is also locally uniformly ψ_1 -integrating. We have the following characterizations of the two concepts.

- (i) It is straightforward to verify from the definition that a set $\mathcal{N} \subset \mathcal{N}_1^1$ is locally uniformly ψ_1 -integrating if and only if every sequence $(\mu_n) \in \mathcal{N}^{\mathbb{N}}$ that converges weakly in \mathcal{N} is uniformly ψ_1 -integrating.
- (ii) The de la Vallée-Poussin theorem (Theorem II.T22 in [19]) implies that a set $\mathcal{N} \subset \mathcal{N}_1^1$ is uniformly ψ_1 -integrating if and only if there exists a sequence $(a_k) \in \mathbb{R}_+^{\mathbb{N}}$ such that $a_k/k \rightarrow \infty$ as $k \rightarrow \infty$ and $\sup_{\mu \in \mathcal{N}} \sum_{k=0}^{\infty} a_k \mu[\{k\}] < \infty$. This implies that a uniformly ψ_1 -integrating set \mathcal{N} is mean bounded in the sense that $\sup_{\mu \in \mathcal{N}} m_{\mu} < \infty$. On the other hand an arbitrary set \mathcal{N} that is “ p th moment bounded” for some $p > 1$ is uniformly ψ_1 -integrating. In particular, a set \mathcal{N} is uniformly ψ_1 -integrating if its elements are supported by a common finite set. \diamond

We may now formulate our main result.

Theorem 2.4 Let $\mathcal{N} \subset \mathcal{N}_1^1$ be such that $m_{\mu} > 1$ for all $\mu \in \mathcal{N}$. Then the following assertions hold:

- (i) The sequence (\widehat{m}_n) is robust on \mathcal{N} if \mathcal{N} is locally uniformly ψ_1 -integrating.
- (ii) The sequence (\widehat{m}_n) is uniformly robust on \mathcal{N} if \mathcal{N} is uniformly ψ_1 -integrating and $\inf_{\mu \in \mathcal{N}} m_{\mu} > 1$.
- (iii) The sequence (\widehat{m}_n) is not robust on \mathcal{N} if the mapping $\mathcal{N} \ni \mu \mapsto m_{\mu}$ is not $(d_{\text{TV}}, |\cdot|)$ -continuous on all of \mathcal{N} .

An outline of the proof is given at the end of this section. The detailed arguments are presented in Sections 3–5.

Remark 2.5 We note that the statement of the theorem remains the same if we consider a Galton-Watson branching process (Z_n) that is started with $z_0 \in \mathbb{N}$ individuals instead of started with 1 individual. The modifications that are needed in the proofs in order to show this slightly more general statement are outlined in Section 6. \diamond

In what follows we give a number of illustrative examples.

Example 2.6 Let us consider the set \mathcal{N}_{bin} of all probability measures that are supported by the set $\{0, 2\}$. Note that each element μ of \mathcal{N}_{bin} corresponds to a Galton-Watson process with binary branching. The set \mathcal{N}_{bin} is obviously uniformly ψ_1 -integrating, such that by part (ii) of Theorem 2.4 the sequence (\hat{m}_n) of Lotka-Nagaev estimators is uniformly robust on \mathcal{N}_{bin} .

Note that an element μ of \mathcal{N}_{bin} is uniquely determined by the probability $p := \mu[\{2\}]$ for 2 offspring. Also note that the total variation distance of two elements μ_1 and μ_2 of \mathcal{N}_{bin} equals the distance of $p_1 := \mu_1[\{2\}]$ and $p_2 := \mu_2[\{2\}]$, i.e. $d_{\text{TV}}(\mu_1, \mu_2) = |p_1 - p_2|$. Thus uniform robustness of the sequence (\hat{m}_n) on \mathcal{N}_{bin} means that for every $\varepsilon > 0$ there is some $\delta > 0$ such that for arbitrary $n \in \mathbb{N}$ and $p_1, p_2 \in [0, 1]$ with $|p_1 - p_2| \leq \delta$ the distributions of the Lotka-Nagaev estimator \hat{m}_n under the parameters p_1 and p_2 are within a Prohorov-distance of ε of one another. Of course, the same holds true for the distributions of the plug-in estimators $\hat{p}^{(n)} = \hat{m}_n/2$.

For applications this becomes relevant if we want to estimate the true parameter p_1 in the \mathcal{N}_{bin} model, but are only able to take observations from a slightly perturbed model with parameter $p_2 \approx p_1$. The above result then tells us that our estimator has “essentially the same” distributional properties as it would have with observations from the true model. \diamond

Example 2.7 Suppose that we would like to estimate p in the model \mathcal{N}_{bin} of the previous example, but in reality the offspring distribution lies in a larger class $\mathcal{N} \supset \mathcal{N}_{\text{bin}}$, i.e. our model is misspecified. As a simple example suppose that \mathcal{N} is the set of all probability measures with support $\{0, 2, 3\}$. Then \mathcal{N} is of course still uniformly ψ_1 -integrating. Note that the total variation distance between an element $\mu_1 \in \mathcal{N}$ with mass $q > 0$ at 3 and an element $\mu_2 \in \mathcal{N}_{\text{bin}}$ that distributes this additional mass among 0 and 2 is exactly q .

The uniform robustness property obtained by Theorem 2.4(ii) tells us then essentially that for q small, i.e. if the model is only slightly misspecified, the distribution of \hat{m}_n (and hence of $\hat{p}^{(n)}$) is still close to the distribution we would have obtained if our model assumption had been correct. \diamond

Example 2.8 The class $\mathcal{N}_{\text{pois}}$ of Poisson distributions Π_λ , $\lambda > 0$, is locally uniformly ψ_1 -integrating by Remark 2.3(i). Indeed, if (Π_{λ_n}) is a sequence in $\mathcal{N}_{\text{pois}}$ such that

$\Pi_{\lambda_n} \rightarrow \Pi_\lambda$ weakly for some $\lambda > 0$, we have in particular that $\lambda_n = -\log(\Pi_{\lambda_n}[\{0\}]) \rightarrow -\log(\Pi_\lambda[\{0\}]) = \lambda$, i.e. convergence of the means. By Theorem 2.20 in [23] this implies that (Π_{λ_n}) is uniformly ψ_1 -integrating. (Note that in the definition of asymptotic uniform integrability on page 17 in [23] “lim sup” can be replaced by “sup”.)

Again we can argue along similar lines as in Example 2.6. If we want to estimate some true λ_1 , but can observe only from a perturbed model with parameter $\lambda_2 \approx \lambda_1$, the robustness still tells us that the distribution of the estimator $\hat{\lambda}^{(n)} = \hat{m}_n$ changes only slightly. However, the influence of the perturbation on this change may now vitally depend on λ_1 because the robustness is not uniform. \diamond

Example 2.9 Consider the class $\mathcal{N}_{\text{poly}}$ of polynomial distributions P_p with existing expectations, i.e. $P_p[\{k\}] = c_p(k+1)^{-p}$, where $p > 2$ and c_p is a normalizing constant. If (P_{p_n}) is a sequence in $\mathcal{N}_{\text{poly}}$ such that $P_{p_n} \rightarrow P_p$ weakly for some $p > 2$, we have by $P_{p_n}[\{k\}] \rightarrow P_p[\{k\}]$ for $k = 0, 1$ that $c_{p_n} \rightarrow c_p$ and $p_n \rightarrow p$ as $n \rightarrow \infty$. Writing $p_* = \inf_n p_n > 2$ and $p^* = \sup_n p_n < \infty$, we obtain

$$\sup_n \sum_{k=\ell}^{\infty} k c_{p_n} (k+1)^{-p_n} \leq \sum_{k=\ell}^{\infty} k c_{p^*} (k+1)^{-p^*} \rightarrow 0 \quad \text{as } \ell \rightarrow \infty.$$

Thus, again by Remark 2.3(i), we see that $\mathcal{N}_{\text{poly}}$ is locally uniformly ψ_1 -integrating. \diamond

As a corollary of Theorem 2.4 we may show that (\hat{m}_n) is robust on \mathcal{N} if and only if \mathcal{N} is locally uniformly ψ_1 -integrating. Recall that the ψ_1 -weak topology on \mathcal{N}_1^1 is defined to be the coarsest topology for which all mappings $\mu \mapsto \int f d\mu$, $f \in \mathbb{F}^1$, are continuous, where \mathbb{F}^1 is the set of all maps $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ with $|f(k)| \leq C_f(1+|k|) = C_f\psi_1(k)$ for all $k \in \mathbb{N}_0$ and some finite constant $C_f > 0$; see, for instance, Section A.5 in [10]. Of course, the ψ_1 -weak topology is finer than the weak topology. On the other hand, it was shown (in a more general setting) in Section 3.1 in [24] that locally uniformly ψ_1 -integrating sets are exactly those subsets of \mathcal{N}_1^1 on which the relative weak topology and the relative ψ_1 -weak topology coincide.

Corollary 2.10 *Let $\mathcal{N} \subset \mathcal{N}_1^1$ be such that $m_\mu > 1$ for all $\mu \in \mathcal{N}$. Then the sequence (\hat{m}_n) is robust on \mathcal{N} if and only if \mathcal{N} is locally uniformly ψ_1 -integrating.*

Proof By part (i) of Theorem 2.4 we know that the sequence (\hat{m}_n) is robust on \mathcal{N} if \mathcal{N} is locally uniformly ψ_1 -integrating.

Now assume that the sequence (\hat{m}_n) is robust on \mathcal{N} . By part (iii) of Theorem 2.4 it follows that the mapping $\mathcal{N} \ni \mu \mapsto m_\mu$ is $(d_{\text{TV}}, |\cdot|)$ -continuous and thus continuous with respect to the weak topology on \mathcal{N} . Suppose that \mathcal{N} is not locally uniformly ψ_1 -integrating. This implies that the relative ψ_1 -weak topology on \mathcal{N} is (strictly) finer than

the relative weak topology on \mathcal{N} . In particular, we can find some $\mu, \mu_1, \mu_2, \dots \in \mathcal{N}$ such that $\mu_n \rightarrow \mu$ weakly but $\mu_n \not\rightarrow \mu$ ψ_1 -weakly. It is easily seen that $\mu_n \rightarrow \mu$ ψ_1 -weakly if and only if $\mu_n \rightarrow \mu$ weakly and $m_{\mu_n} \rightarrow m_\mu$. So we obtain $m_{\mu_n} \not\rightarrow m_\mu$. This contradicts the weak continuity of $\mu \mapsto m_\mu$ on \mathcal{N} . \square

We finish this section by giving an outline of the proof of Theorem 2.4(i). The proof strategy for part (ii) is exactly the same and the proof of part (iii) is based on a simple contradiction argument; see Theorem 5.5.

As mentioned in the introduction robustness of a sequence (\widehat{m}_n) on \mathcal{N} means equicontinuity of the set of maps $\{\mathcal{N} \rightarrow \mathcal{M}_1^+, \mu \mapsto \mathbb{P}^\mu \circ \widehat{m}_n^{-1} : n \in \mathbb{N}\}$. In Section 5 we show this equicontinuity by separately showing continuity (“finite sample robustness”) and asymptotic equicontinuity (“asymptotic robustness”) of these maps.

Finite sample robustness is shown in Theorem 5.4 by a coupling argument using Strassen’s theorem and the fact that close offspring distributions generate close distributions of pairs (Z_{n-1}, Z_n) of generation sizes for any n (Lemma 3.5).

Asymptotic robustness is a somewhat more involved matter. In Lemma 5.2 we first show that it is enough to prove asymptotic robustness if for each \widehat{m}_n we condition on non-extinction up to time $n - 1$. The required asymptotic closeness of the conditional distributions of \widehat{m}_n given $Z_{n-1} > 0$, uniformly over μ_2 from a δ -ball of offspring distributions around each $\mu_1 \in \mathcal{N}$, is then proved by using the locally uniform conditional weak consistency property of (\widehat{m}_n) (Theorem 4.1) and noting that the remaining distance between m_{μ_1} and m_{μ_2} is small (Lemma 3.1).

The detailed arguments can be found in the following sections. We start with a series of general probabilistic lemmas on Galton–Watson processes in Section 3. In Section 4 we show [locally] uniform weak consistency of the Lotka–Nagaev estimator on [locally] uniformly ψ_1 -integrating sets, conditional on non-extinction. After these preparation we carry out the proof of Theorem 2.4 in Section 5. Finally, in Section 6 we summarize the modifications necessary to see that Theorem 2.4 also holds for Galton–Watson processes with general initial states.

3 Auxiliary lemmas about Galton–Watson processes

Lemma 3.1 (i) *Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a locally uniformly ψ_1 -integrating set. Then the mapping $\mathcal{N} \ni \mu \mapsto m_\mu$ is $(d_{\text{TV}}, |\cdot|)$ -continuous.*

(ii) *If $\mathcal{N} \subset \mathcal{N}_1^1$ is even uniformly ψ_1 -integrating, then the mapping $\mathcal{N} \ni \mu \mapsto m_\mu$ is uniformly $(d_{\text{TV}}, |\cdot|)$ -continuous.*

Proof We first prove part (i). Fix $\varepsilon > 0$ and $\mu_1 \in \mathcal{N}$. Since \mathcal{N} was assumed to be locally uniformly ψ_1 -integrating, we can find some $\delta > 0$ and $\ell_\varepsilon \in \mathbb{N}$ such that for every

$\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$, we have $\sum_{k=\ell_\varepsilon}^{\infty} k \mu_2[\{k\}] < \varepsilon/4$. It follows that

$$\begin{aligned} |m_{\mu_1} - m_{\mu_2}| &\leq \sum_{k=1}^{\infty} k |\mu_1[\{k\}] - \mu_2[\{k\}]| \\ &\leq \ell_\varepsilon \sum_{k=1}^{\ell_\varepsilon} |\mu_1[\{k\}] - \mu_2[\{k\}]| + \sum_{k=\ell_\varepsilon+1}^{\infty} k |\mu_1[\{k\}] - \mu_2[\{k\}]| \\ &\leq \ell_\varepsilon 2 d_{\text{TV}}(\mu_1, \mu_2) + \varepsilon/2. \end{aligned}$$

Thus, choosing $\delta_\varepsilon := \min\{\delta; \ell_\varepsilon^{-1}\varepsilon/4\}$ we have that $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_\varepsilon$ implies $|m_{\mu_1} - m_{\mu_2}| \leq \varepsilon$. This completes the proof of part (i).

Part (ii) can be shown analogously. Set (informally) $\delta := \infty$ and note that ℓ_ε can be chosen independently of μ_1 when \mathcal{N} is uniformly ψ_1 -integrating. \square

Let us fix some more notation regarding the Galton–Watson process. We let

$$f_\mu(s) := \sum_{k \in \mathbb{N}_0} s^k \mu[\{k\}], \quad 0 \leq s \leq 1,$$

be the generating function of the offspring distribution μ . We also use $f_\mu^{(n)}$ to denote the n th iterate of f_μ , which is the generating function of Z_n (recall that $Z_0 = 1$). By q_μ we denote the extinction probability of the associated Galton–Watson branching process, that is,

$$q_\mu := \mathbb{P}^\mu[Z_n = 0 \text{ for some } n \in \mathbb{N}].$$

Except for some of the lemmas in the present section, we assume in this article that $m_\mu > 1$. Recall that q_μ is then the unique solution of $f_\mu(s) = s$ in $s \in [0, 1)$. The generating function f_μ is strictly increasing and strictly convex, which implies $f'_\mu(q_\mu) < 1$. Furthermore we have $f_\mu^{(n)}(s) \nearrow q_\mu$ as $n \rightarrow \infty$ for every $s \in [0, q)$. See [2], Section I.3 and I.5, for this and similar basic results.

Lemma 3.2 (i) *Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a locally uniformly ψ_1 -integrating set with $m_\mu > 1$ for all $\mu \in \mathcal{N}$. Then for every $\mu_1 \in \mathcal{N}$ there exist some $p > 0$ and $\delta > 0$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$,*

$$q_{\mu_2} \leq 1 - p, \tag{6}$$

$$f'_{\mu_2}(q_{\mu_2}) \leq 1 - p. \tag{7}$$

(ii) *If $\mathcal{N} \subset \mathcal{N}_1^1$ is even uniformly ψ_1 -integrating with $\inf_{\mu \in \mathcal{N}} m_\mu > 1$, then there exists a $p > 0$ such that*

$$\sup_{\mu \in \mathcal{N}} q_\mu \leq 1 - p, \tag{8}$$

$$\sup_{\mu \in \mathcal{N}} f'_\mu(q_\mu) \leq 1 - p. \tag{9}$$

Proof We first prove part (i). Let $\mu_1 \in \mathcal{N}$. We start by showing a locally uniform continuity of f'_μ at 1 and μ_1 , meaning that for all $\varepsilon > 0$ there exist some $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$,

$$|f'_{\mu_2}(1) - f'_{\mu_2}(s)| \leq \varepsilon \quad \text{for all } s \in [1 - \delta_2, 1]. \quad (10)$$

Indeed, by the assumption on \mathcal{N} we can choose for fixed $\varepsilon > 0$ some $\delta_1 = \delta_1(\varepsilon) > 0$ and $\ell = \ell(\varepsilon) \in \mathbb{N}$ such that $\sum_{k=\ell+1}^{\infty} k \mu_2[\{k\}] \leq \varepsilon/4$ for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$. Set $\delta_2 = \delta_2(\varepsilon) := \frac{\varepsilon}{2\ell^2}$. Then we have for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$ and all $s \in [1 - \delta_2, 1]$,

$$\begin{aligned} |f'_{\mu_2}(1) - f'_{\mu_2}(s)| &= \left| \sum_{k=1}^{\infty} k(1 - s^{k-1}) \mu_2[\{k\}] \right| \\ &\leq \sum_{k=1}^{\ell} k(k-1)(1-s) \mu_2[\{k\}] + 2 \sum_{k=\ell+1}^{\infty} k \mu_2[\{k\}] \\ &\leq \ell^2 \delta_2 + 2 \frac{\varepsilon}{4} \\ &= \varepsilon, \end{aligned}$$

where we have used that $1 - s^{k-1} \leq (k-1)(1-s)$ for $s \in [0, 1]$. This shows (10).

Next, recall that $m_{\mu_1} > 1$ and choose $\varepsilon > 0$ small enough such that $2\varepsilon < m_{\mu_1} - 1$. By Lemma 3.1 we can find some $\delta_3 = \delta_3(\varepsilon) > 0$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_3$,

$$|m_{\mu_1} - m_{\mu_2}| \leq \varepsilon. \quad (11)$$

Now we use (10) and (11) in order to obtain some $\delta_1 \in (0, \delta_3]$ and $\delta_2 \in (0, \delta_3]$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$ and all $s \in [1 - \delta_2, 1]$,

$$\begin{aligned} f'_{\mu_2}(s) &= f'_{\mu_2}(1) - (f'_{\mu_2}(1) - f'_{\mu_2}(s)) \\ &= m_{\mu_2} - (f'_{\mu_2}(1) - f'_{\mu_2}(s)) \\ &\geq m_{\mu_1} - 2\varepsilon > 1. \end{aligned} \quad (12)$$

From this we get in particular that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$,

$$f_{\mu_2}(s) \leq 1 - (1-s)(m_{\mu_1} - 2\varepsilon) < s \quad \text{for all } s \in [1 - \delta_2, 1].$$

Since $q_{\mu_2} < 1$ and $f_{\mu_2}(q_{\mu_2}) = q_{\mu_2}$, this implies that $q_{\mu_2} < 1 - \delta_2$ for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$, which shows (6) with $p := \delta_2$ and $\delta := \delta_1$. Also, using the convexity of f_{μ_2} and the fact that $f_{\mu_2}(1 - \delta_2) \leq 1 - \delta_2(m_{\mu_1} - 2\varepsilon)$ it is easy to see that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta_1$

$$f'_{\mu_2}(q_{\mu_2}) \leq \frac{f_{\mu_2}(1 - \delta_2) - f_{\mu_2}(0)}{1 - \delta_2} \leq \frac{1 - \delta_2(m_{\mu_1} - 2\varepsilon)}{1 - \delta_2} < 1,$$

where we have bounded the left hand side by the slope of the line connecting $(0, 0)$ with $(1 - \delta_2, 1 - \delta_2(m_{\mu_1} - 2\varepsilon))$. This shows (7) with $p := 1 - (1 - \delta_2(m_{\mu_1} - 2\varepsilon))/(1 - \delta_2)$ and $\delta := \delta_1$, and completes the proof of part (i).

Part (ii) can be shown analogously. Set (informally) $\delta_1 := \delta_3 := \infty$, skip (11), and replace m_{μ_1} by $\underline{m} := \inf_{\mu \in \mathcal{N}} m_\mu > 1$ in what follows. \square

Lemma 3.3 (i) *Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a locally uniformly ψ_1 -integrating set with $m_\mu > 1$ for all $\mu \in \mathcal{N}$. Then for every $\mu_1 \in \mathcal{N}$, $k \in \mathbb{N}$, $\varepsilon > 0$ there exist some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \mathbb{P}^{\mu_2}[Z_n = k | Z_n > 0] \leq \varepsilon \quad \text{for all } n \geq n_0. \quad (13)$$

(ii) *If $\mathcal{N} \subset \mathcal{N}_1^1$ is even uniformly ψ_1 -integrating with $\inf_{\mu \in \mathcal{N}} m_\mu > 1$, then for every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that*

$$\sup_{\mu \in \mathcal{N}} \mathbb{P}^\mu[Z_n = k | Z_n > 0] \leq \varepsilon \quad \text{for all } n \geq n_0. \quad (14)$$

Proof We first prove part (i). Fix $\mu_1 \in \mathcal{N}$, $k \in \mathbb{N}$, and $\varepsilon > 0$. Let $p \in (0, 1)$ and $\delta > 0$ be as in part (i) of Lemma 3.2, and $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$. Let A be the event that a Galton–Watson branching process survives and let B be the event that it goes extinct. We have by $\mathbb{P}^\mu[Z_n > 0] \geq 1 - q_\mu$ that

$$\begin{aligned} \mathbb{P}^{\mu_2}[Z_n = k | Z_n > 0] &= \mathbb{P}^{\mu_2}[\{Z_n = k\} \cap A | Z_n > 0] + \mathbb{P}^{\mu_2}[\{Z_n = k\} \cap B | Z_n > 0] \\ &\leq \frac{\mathbb{P}^{\mu_2}[\{Z_n = k\} \cap A \cap \{Z_n > 0\}]}{\mathbb{P}^{\mu_2}[Z_n > 0]} + \mathbb{P}^{\mu_2}[B | Z_n > 0] \\ &\leq \frac{\mathbb{P}^{\mu_2}[\{Z_n = k\} \cap A]}{1 - q_{\mu_2}} + \mathbb{P}^{\mu_2}[B | Z_n > 0] \\ &= \mathbb{P}^{\mu_2}[Z_n = k | A] + \mathbb{P}^{\mu_2}[B | Z_n > 0]. \end{aligned} \quad (15)$$

For bounding the first term we decompose $Z_n = Z_n^{(1)} + Z_n^{(2)}$ where $Z_n^{(1)}$ is the number of particles among Z_n with infinite line of descent. We then use the fact that $Z_n^{(1)}$ under $\mathbb{P}^{\mu_2}[\cdot | A]$ has the same distribution as Z_n under $\mathbb{P}^{\hat{\mu}_2}$ where $\hat{\mu}_2$ is an offspring distribution with generating function

$$f_{\hat{\mu}_2}(s) = \frac{f_{\mu_2}((1 - q_{\mu_2})s + q_{\mu_2}) - q_{\mu_2}}{1 - q_{\mu_2}}, \quad (16)$$

see Theorem I.12.1 of [2]. Note that $f_{\hat{\mu}_2}$ results from taking f_{μ_2} on the square $[q_{\mu_2}, 1]^2$ and stretching it linearly to the unit square $[0, 1]^2$. Naturally, we have that the corresponding Galton–Watson branching process is supercritical with $\hat{\mu}_2[\{0\}] = f_{\hat{\mu}_2}(0) = 0$ and so also $q_{\hat{\mu}_2} = 0$. By (7) of Lemma 3.2 and the choice of p ,

$$\hat{\mu}_2[\{1\}] = f'_{\hat{\mu}_2}(0) = f'_{\mu_2}(q_{\mu_2}) \leq 1 - p. \quad (17)$$

Under $\mathbb{P}^{\hat{\mu}_2}$, the process Z_n is a.s. increasing in n . The probability that it increases by a positive quantity is at least $1 - f'_{\mu_2}(q_{\mu_2}) \geq p$. Thus, if $B_{n,p}$ denotes the binomial distribution with parameters n and p we have

$$\begin{aligned}
\mathbb{P}^{\mu_2}[Z_n = k|A] &\leq \mathbb{P}^{\mu_2}[Z_n \leq k|A] \\
&\leq \mathbb{P}^{\mu_2}[Z_n^{(1)} \leq k|A] \\
&= \mathbb{P}^{\hat{\mu}_2}[Z_n \leq k] \\
&\leq B_{n,p}[\{0, \dots, k\}] \\
&\leq \varepsilon/2
\end{aligned} \tag{18}$$

for all $n \geq n_1$ for some sufficiently large $n_1 \in \mathbb{N}$.

It remains to bound the probability of extinction given that $Z_n > 0$. Here, we rewrite

$$\mathbb{P}^{\mu_2}[B|Z_n > 0] = \frac{\mathbb{P}^{\mu_2}[Z_n > 0|B] \cdot \mathbb{P}^{\mu_2}[B]}{\mathbb{P}^{\mu_2}[Z_n > 0]} \leq \mathbb{P}^{\mu_2}[Z_n > 0|B] \frac{q_{\mu_2}}{1 - q_{\mu_2}}.$$

Due to (6) of Lemma 3.2 it then remains to bound $\mathbb{P}^{\mu_2}[Z_n > 0|B]$ uniformly. Here, we use the fact that Z_n is under $\mathbb{P}^{\mu_2}[\cdot|B]$ a subcritical Galton–Watson branching process with offspring distribution μ_2^* described via its generating function

$$f_{\mu_2^*}(s) = \frac{1}{q_{\mu_2}} f_{\mu_2}(sq_{\mu_2}),$$

see Theorem I.12.3 of [2]. Therefore, we have $m_{\mu_2^*} = f'_{\mu_2^*}(1) = f'_{\mu_2}(q_{\mu_2}) \leq 1 - p$ by (7) of Lemma 3.2 and the choice of p . Thus, by Markov's inequality

$$\mathbb{P}^{\mu_2}[Z_n > 0|B] = \mathbb{P}^{\mu_2^*}[Z_n > 0] = \mathbb{P}^{\mu_2^*}[Z_n \geq 1] \leq \mathbb{E}^{\mu_2^*}[Z_n] = m_{\mu_2^*}^n \leq (1 - p)^n \leq \varepsilon/2$$

for all $n \geq n_0$ for some sufficiently large $n_0 \geq n_1$. This completes the proof of part (i).

Part (ii) can be shown analogously, using (8)–(9) instead of (6)–(7). \square

Lemma 3.4 (i) For every $\mu_1 \in \mathcal{N}_1^1$ with $m_{\mu_1} > 1$ and every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$\begin{aligned}
\mu_2 \in \mathcal{N}_1^1, \quad d_{\text{TV}}(\mu_1, \mu_2) &\leq \delta \\
\implies |\mathbb{P}^{\mu_1}[Z_n = 0] - \mathbb{P}^{\mu_2}[Z_n = 0]| &\leq \varepsilon \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{19}$$

(ii) Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a uniformly ψ_1 -integrating set with $\inf_{\mu \in \mathcal{N}} m_{\mu} > 1$. Then for every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$\begin{aligned}
\mu_1, \mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) &\leq \delta \\
\implies |\mathbb{P}^{\mu_1}[Z_n = 0] - \mathbb{P}^{\mu_2}[Z_n = 0]| &\leq \varepsilon \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{20}$$

Proof First note that for any $\mu_1, \mu_2 \in \mathcal{N}_1^1$, we have

$$\begin{aligned}
& |f_{\mu_1}(s) - f_{\mu_2}(s)| \\
&= \left| \sum_{k \in \mathbb{N}_0} s^k (\mu_1[\{k\}] - \mu_2[\{k\}]) \right| \\
&\leq \max \left\{ \sum_{\substack{k \in \mathbb{N}_0 \\ \mu_1(k) > \mu_2(k)}} s^k (\mu_1(k) - \mu_2(k)), \sum_{\substack{k \in \mathbb{N}_0 \\ \mu_1(k) < \mu_2(k)}} s^k (\mu_2(k) - \mu_1(k)) \right\} \\
&\leq d_{\text{TV}}(\mu_1, \mu_2)
\end{aligned} \tag{21}$$

by the fact that

$$\sum_{\substack{k \in \mathbb{N}_0 \\ \mu_1(k) > \mu_2(k)}} (\mu_1(k) - \mu_2(k)) = \sum_{\substack{k \in \mathbb{N}_0 \\ \mu_1(k) < \mu_2(k)}} (\mu_2(k) - \mu_1(k)) = d_{\text{TV}}(\mu_1, \mu_2).$$

We now show part (i). Let $\varepsilon > 0$ and $\mu_1 \in \mathcal{N}_1^1$ with $m_{\mu_1} > 1$. Since $f'_{\mu_1}(q_{\mu_1}) < 1$ and f'_{μ_1} is continuous, we may choose $\bar{q} > q_{\mu_1}$ such that $\bar{\gamma} := (f_{\mu_1})'(\bar{q}) < 1$. Set

$$\delta := \min\{(\bar{q} - f_{\mu_1}(\bar{q}))/2; (1 - \bar{\gamma})\varepsilon\} > 0. \tag{22}$$

Letting $\mu_2 \in \mathcal{N}_1^1$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$, we obtain by (21), (22) and $f_{\mu_1}(\bar{q}) < \bar{q}$ that

$$f_{\mu_2}(\bar{q}) \leq f_{\mu_1}(\bar{q}) + |f_{\mu_2}(\bar{q}) - f_{\mu_1}(\bar{q})| \leq f_{\mu_1}(\bar{q}) + \frac{\bar{q} - f_{\mu_1}(\bar{q})}{2} < \bar{q}.$$

Since $f_{\mu_2}(s) < s$ holds if and only if $s > q_{\mu_2}$, we conclude $q_{\mu_2} < \bar{q}$. Note that $0 \leq f_{\mu_i}^{(1)}(0) \leq f_{\mu_i}^{(2)}(0) \leq \dots \leq q_{\mu_i} \leq \bar{q}$, $i = 1, 2$. Furthermore, since f_{μ_1} is convex, it is Lipschitz continuous on $[0, \bar{q}]$ with constant $\bar{\gamma} < 1$. Therefore we have for $n \geq 2$

$$\begin{aligned}
& |f_{\mu_1}^{(n)}(0) - f_{\mu_2}^{(n)}(0)| \\
&\leq |f_{\mu_1}(f_{\mu_1}^{(n-1)}(0)) - f_{\mu_1}(f_{\mu_2}^{(n-1)}(0))| + |f_{\mu_1}(f_{\mu_2}^{(n-1)}(0)) - f_{\mu_2}(f_{\mu_2}^{(n-1)}(0))| \\
&\leq \bar{\gamma} |f_{\mu_1}^{(n-1)}(0) - f_{\mu_2}^{(n-1)}(0)| + d_{\text{TV}}(\mu_1, \mu_2).
\end{aligned} \tag{23}$$

For the case $n = 1$ we obtain by (21) that

$$|f_{\mu_1}^{(1)}(0) - f_{\mu_2}^{(1)}(0)| = |f_{\mu_1}(0) - f_{\mu_2}(0)| \leq d_{\text{TV}}(\mu_1, \mu_2).$$

By induction we obtain from this and inequality (23) that

$$|\mathbb{P}^{\mu_1}[Z_n = 0] - \mathbb{P}^{\mu_2}[Z_n = 0]| = |f_{\mu_1}^{(n)}(0) - f_{\mu_2}^{(n)}(0)| \leq \left(\sum_{k=0}^n \bar{\gamma}^k \right) d_{\text{TV}}(\mu_1, \mu_2) \leq \varepsilon \tag{24}$$

for all $n \in \mathbb{N}$. This completes the proof of part (i).

Part (ii) can be shown in a similar way. Set $\delta := (1 - \gamma^*)\varepsilon$, where $\gamma^* := \sup_{\mu \in \mathcal{N}} f'_\mu(q_\mu) < 1$ by Lemma 3.2(ii). Let $\mu_1, \mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$ and set $q^* := \max(q_{\mu_1}, q_{\mu_2})$. By convexity the function f_{μ_i} is Lipschitz continuous on $[0, q^*]$ with constant $f'_{\mu_i}(q^*)$ for $i = 1, 2$. Hence using that $f_{\mu_i}^{(n)}(0) \leq q_{\mu_i} \leq q^*$ inequality (23) can be replaced by

$$|f_{\mu_1}^{(n)}(0) - f_{\mu_2}^{(n)}(0)| \leq f'_{\mu_i}(q^*) |f_{\mu_1}^{(n-1)}(0) - f_{\mu_2}^{(n-1)}(0)| + d_{\text{TV}}(\mu_1, \mu_2) \quad \text{for } i = 1, 2. \quad (25)$$

Since $\min_{i \in \{1, 2\}} f'_{\mu_i}(q^*) \leq \gamma^*$, we obtain that inequality (24) holds for all $n \in \mathbb{N}$ with $\bar{\gamma}$ replaced by γ^* . \square

Now, let $\mathcal{N}_1^{1,n}$ be the set of all probability measures on \mathbb{N}_0^n with marginal distributions in \mathcal{N}_1^1 and $d_{\text{TV}}^{(n)}$ the total variation distance on $\mathcal{N}_1^{1,n}$. The following lemma shows in particular that the mapping $\mathcal{N}_1^1 \rightarrow \mathcal{N}_1^{1,n}$, $\mu \mapsto \mathbb{P}^\mu \circ (Z_1, \dots, Z_n)^{-1}$ is $(d_{\text{TV}}, d_{\text{TV}}^{(n)})$ -continuous.

Lemma 3.5 *For every $\mu_1, \mu_2 \in \mathcal{N}_1^1$ and $n \in \mathbb{N}$ we have*

$$d_{\text{TV}}^{(n)}(\mathbb{P}^{\mu_1} \circ (Z_1, \dots, Z_n)^{-1}, \mathbb{P}^{\mu_2} \circ (Z_1, \dots, Z_n)^{-1}) \leq C_n(\mu_1, \mu_2) d_{\text{TV}}(\mu_1, \mu_2), \quad (26)$$

where $C_n(\mu_1, \mu_2) := \min\{\sum_{i=1}^n m_{\mu_1}^{i-1}, \sum_{i=1}^n m_{\mu_2}^{i-1}\}$.

Proof Let $\mu_1, \mu_2 \in \mathcal{N}_1^1$, $n \in \mathbb{N}$, and $(k_1, \dots, k_n) \in \mathbb{N}_0^n$. By the Markov property we have

$$\begin{aligned} \mathbb{P}^{\mu_i}[(Z_1, \dots, Z_n) = (k_1, \dots, k_n)] &= \mathbb{P}^{\mu_i}[Z_1 = k_1] \cdot \mathbb{P}^{\mu_i}[Z_2 = k_2 | Z_1 = k_1] \cdots \mathbb{P}^{\mu_i}[Z_n = k_n | Z_{n-1} = k_{n-1}] \\ &= \prod_{j=1}^n \mu_i^{*k_{j-1}}[\{k_j\}] \end{aligned} \quad (27)$$

for $i = 1, 2$, where we set $k_0 := 1$. Here μ_i^{*k} denotes the k th convolution of the measure μ_i and we set $\mu_i^{*1} := \mu_i$. Note furthermore that for $x_j, y_j \geq 0$,

$$\begin{aligned} \left| \prod_{j=1}^n x_j - \prod_{j=1}^n y_j \right| &= \left| \sum_{i=1}^n \left[\left(\prod_{j=1}^{i-1} y_j \right) x_i \left(\prod_{\ell=i+1}^n x_\ell \right) - \left(\prod_{j=1}^{i-1} y_j \right) y_i \left(\prod_{\ell=i+1}^n x_\ell \right) \right] \right| \\ &\leq \sum_{i=1}^n |x_i - y_i| \prod_{j=1}^{i-1} y_j \prod_{\ell=i+1}^n x_\ell. \end{aligned} \quad (28)$$

Combining (27) and (28) we obtain

$$\begin{aligned}
& 2 d_{\text{TV}}^{(n)}(\mathbb{P}^{\mu_1} \circ (Z_1, \dots, Z_n)^{-1}, \mathbb{P}^{\mu_2} \circ (Z_1, \dots, Z_n)^{-1}) \\
&= \sum_{(k_1, \dots, k_n) \in \mathbb{N}_0^n} \left| \mathbb{P}^{\mu_1}[(Z_1, \dots, Z_n) = (k_1, \dots, k_n)] - \mathbb{P}^{\mu_2}[(Z_1, \dots, Z_n) = (k_1, \dots, k_n)] \right| \\
&= \sum_{(k_1, \dots, k_n) \in \mathbb{N}_0^n} \left| \prod_{j=1}^n \mu_1^{*k_{j-1}}[\{k_j\}] - \prod_{j=1}^n \mu_2^{*k_{j-1}}[\{k_j\}] \right| \\
&\leq \sum_{(k_1, \dots, k_n) \in \mathbb{N}_0^n} \sum_{i=1}^n \left| \mu_1^{*k_{i-1}}[\{k_i\}] - \mu_2^{*k_{i-1}}[\{k_i\}] \right| \prod_{j=1}^{i-1} \mu_2^{*k_{j-1}}[\{k_j\}] \prod_{\ell=i+1}^n \mu_1^{*k_{\ell-1}}[\{k_\ell\}] \\
&= \sum_{i=1}^n \left\{ \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_n \in \mathbb{N}_0} \left| \mu_1^{*k_{i-1}}[\{k_i\}] - \mu_2^{*k_{i-1}}[\{k_i\}] \right| \prod_{j=1}^{i-1} \mu_2^{*k_{j-1}}[\{k_j\}] \prod_{\ell=i+1}^n \mu_1^{*k_{\ell-1}}[\{k_\ell\}] \right\} \\
&=: \sum_{i=1}^n S_i(\mu_1, \mu_2). \tag{29}
\end{aligned}$$

Using $\sum_{k_n \in \mathbb{N}_0} \mu_1^{*k_{n-1}}[\{k_n\}] = 1$, we have for $2 \leq i \leq n-1$ that

$$\begin{aligned}
& S_i(\mu_1, \mu_2) \\
&= \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_{n-1} \in \mathbb{N}_0} \left| \mu_1^{*k_{i-1}}[\{k_i\}] - \mu_2^{*k_{i-1}}[\{k_i\}] \right| \sum_{k_n \in \mathbb{N}_0} \prod_{j=1}^{i-1} \mu_2^{*k_{j-1}}[\{k_j\}] \prod_{\ell=i+1}^n \mu_1^{*k_{\ell-1}}[\{k_\ell\}] \\
&= \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_{n-1} \in \mathbb{N}_0} \left| \mu_1^{*k_{i-1}}[\{k_i\}] - \mu_2^{*k_{i-1}}[\{k_i\}] \right| \prod_{j=1}^{i-1} \mu_2^{*k_{j-1}}[\{k_j\}] \prod_{\ell=i+1}^{n-1} \mu_1^{*k_{\ell-1}}[\{k_\ell\}] \\
&= \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_i \in \mathbb{N}_0} \left| \mu_1^{*k_{i-1}}[\{k_i\}] - \mu_2^{*k_{i-1}}[\{k_i\}] \right| \prod_{j=1}^{i-1} \mu_2^{*k_{j-1}}[\{k_j\}],
\end{aligned}$$

where the last step follows by iteration of the previous two steps. Since $d_{\text{TV}}(\mu_1^{*k}, \mu_2^{*k}) \leq k d_{\text{TV}}(\mu_1, \mu_2)$ for every k , we can proceed as

$$\begin{aligned}
& \leq \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_{i-1} \in \mathbb{N}_0} 2 k_{i-1} d_{\text{TV}}(\mu_1, \mu_2) \prod_{j=1}^{i-1} \mu_2^{*k_{j-1}}[\{k_j\}] \\
& \leq 2 d_{\text{TV}}(\mu_1, \mu_2) \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_{i-1} \in \mathbb{N}_0} k_{i-1} \mu_2^{*k_{i-2}}[\{k_{i-1}\}] \mu_2^{*k_{i-3}}[\{k_{i-2}\}] \cdots \mu_2^{*k_0}[\{k_1\}] \\
& = 2 d_{\text{TV}}(\mu_1, \mu_2) \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_{i-2} \in \mathbb{N}_0} k_{i-2} m_{\mu_2} \mu_2^{*k_{i-3}}[\{k_{i-2}\}] \cdots \mu_2^{*k_0}[\{k_1\}] \\
& = 2 d_{\text{TV}}(\mu_1, \mu_2) m_{\mu_2}^{i-1}, \tag{30}
\end{aligned}$$

where the last step follows by iteration. Note that this is again true for $2 \leq i \leq n-1$ since the expression $i-3$ only appears in the above in order to illustrate the iteration

for larger $i \geq 3$. Analogously we obtain

$$S_1(\mu_1, \mu_2) \leq 2 d_{\text{TV}}(\mu_1, \mu_2) \quad \text{and} \quad S_n(\mu_1, \mu_2) \leq 2 d_{\text{TV}}(\mu_1, \mu_2) m_{\mu_2}^{n-1}. \quad (31)$$

Now, (29)–(31) imply (26) with $C_n(\mu_1, \mu_2)$ replaced by $\sum_{i=1}^n m_{\mu_2}^{i-1}$. Due to symmetry the proof for (26) with $C_n(\mu_1, \mu_2)$ replaced by $\sum_{i=1}^n m_{\mu_1}^{i-1}$ is analogous, which shows (26). \square

Lemma 3.6 (i) *Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a locally uniformly ψ_1 -integrating set. Then for every $\mu_1 \in \mathcal{N}$, $\varepsilon > 0$, and $\eta > 0$ there exist some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$,*

$$\mathbb{P}^{\mu_2} \left[\left| \frac{1}{n} \sum_{i=1}^n X_{0,i} - m_{\mu_2} \right| \geq \eta \right] \leq \varepsilon \quad \text{for all } n \geq n_0. \quad (32)$$

(ii) *If $\mathcal{N} \subset \mathcal{N}_1^1$ is even uniformly ψ_1 -integrating, then for every $\varepsilon > 0$ and $\eta > 0$ there exists some $n_0 \in \mathbb{N}$ such that*

$$\sup_{\mu \in \mathcal{N}} \mathbb{P}^\mu \left[\left| \frac{1}{n} \sum_{i=1}^n X_{0,i} - m_\mu \right| \geq \eta \right] \leq \varepsilon \quad \text{for all } n \geq n_0. \quad (33)$$

Proof Part (ii) is an immediate consequence of Chung's [3] uniform (strong) law of large numbers. So it suffices to prove part (i). Fix $\mu_1 \in \mathcal{N}$, $\varepsilon \in (0, 2)$ and $\eta > 0$. For every $\ell \in \mathbb{N}$ let $X_{0,i}^\ell := X_{0,i} \mathbb{1}_{\{X_{0,i} \leq \ell\}}$ be the ℓ -truncation of $X_{0,i}$. Using the decomposition $X_{0,i} = X_{0,i}^\ell + X_{0,i} \mathbb{1}_{\{X_{0,i} > \ell\}}$ and the triangle inequality, we obtain

$$\begin{aligned} \mathbb{P}^{\mu_2} \left[\left| \frac{1}{n} \sum_{i=1}^n X_{0,i} - m_{\mu_2} \right| \geq \eta \right] &\leq \mathbb{P}^{\mu_2} \left[\left| \frac{1}{n} \sum_{i=1}^n X_{0,i}^\ell - \mathbb{E}^{\mu_2}[X_{0,1}^\ell] \right| \geq \eta/3 \right] \\ &\quad + \mathbb{P}^{\mu_2} \left[\frac{1}{n} \sum_{i=1}^n X_{0,i} \mathbb{1}_{\{X_{0,i} > \ell\}} \geq \eta/3 \right] \\ &\quad + \mathbb{P}^{\mu_2} \left[\mathbb{E}^{\mu_2}[X_{0,1} \mathbb{1}_{\{X_{0,1} > \ell\}}] \geq \eta/3 \right] \\ &=: S_1(\eta, n, \ell, \mu_2) + S_2(\eta, n, \ell, \mu_2) + S_3(\eta, \ell, \mu_2). \end{aligned}$$

By Markov's inequality $S_2(\eta, n, \ell, \mu_2)$ is bounded above by $3\eta^{-1} \mathbb{E}^{\mu_2}[X_{0,1} \mathbb{1}_{\{X_{0,1} > \ell\}}]$. The assumption on \mathcal{N} yields that one can choose $\delta > 0$ and $\ell_0 = \ell_0(\varepsilon, \eta) \in \mathbb{N}$ such that

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \mathbb{E}^{\mu_2}[X_{0,1} \mathbb{1}_{\{X_{0,1} > \ell_0\}}] \leq \varepsilon\eta/6 < \eta/3.$$

Hence

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad S_2(\eta, n, \ell_0, \mu_2) + S_3(\eta, \ell_0, \mu_2) \leq \varepsilon/2 + 0$$

for all $n \in \mathbb{N}$. By Chebychev's inequality, we further obtain (regardless of $\mu_2 \in \mathcal{N}_1^1$)

$$S_1(\eta, n, \ell_0, \mu_2) \leq 9\eta^{-2} \ell_0^2 n^{-1} \leq \varepsilon/2$$

for all $n \geq n_0$ for some sufficiently large $n_0 \in \mathbb{N}$. \square

4 Uniform conditional weak consistency of the Lotka–Nagaev estimator

Theorem 4.1 (i) Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a locally uniformly ψ_1 -integrating set with $m_\mu > 1$ for all $\mu \in \mathcal{N}$. Then for every $\mu_1 \in \mathcal{N}$, $\varepsilon > 0$, and $\eta > 0$ there exist some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$,

$$\mathbb{P}^{\mu_2} [|\hat{m}_n - m_{\mu_2}| \geq \eta \mid Z_{n-1} > 0] \leq \varepsilon \quad \text{for all } n \geq n_0. \quad (34)$$

(ii) If $\mathcal{N} \subset \mathcal{N}_1^1$ is even uniformly ψ_1 -integrating with $\inf_{\mu \in \mathcal{N}} m_\mu > 1$, then for every $\varepsilon > 0$ and $\eta > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$\sup_{\mu \in \mathcal{N}} \mathbb{P}^\mu [|\hat{m}_n - m_\mu| \geq \eta \mid Z_{n-1} > 0] \leq \varepsilon \quad \text{for all } n \geq n_0.$$

Proof We first prove part (i). Fix $\mu_1 \in \mathcal{N}$, $\varepsilon > 0$, and $\eta > 0$. For every $\mu_2 \in \mathcal{N}$ we have

$$\begin{aligned} & \mathbb{P}^{\mu_2} [|\hat{m}_n - m_{\mu_2}| \geq \eta \mid Z_{n-1} > 0] \\ &= \sum_{k=1}^{\infty} \mathbb{P}^{\mu_2} [|\hat{m}_n - m_{\mu_2}| \geq \eta \mid Z_{n-1} = k] \mathbb{P}^{\mu_2} [Z_{n-1} = k \mid Z_{n-1} > 0] \\ &= \sum_{k=1}^{\infty} \mathbb{P}^{\mu_2} [|\bar{Z}_n/k - m_{\mu_2}| \geq \eta \mid Z_{n-1} = k] \mathbb{P}^{\mu_2} [Z_{n-1} = k \mid Z_{n-1} > 0] \\ &= \sum_{k=1}^{\infty} \mathbb{P}^{\mu_2} \left[\left| \frac{1}{k} \sum_{i=1}^k X_{n-1,i} - m_{\mu_2} \right| \geq \eta \right] \mathbb{P}^{\mu_2} [Z_{n-1} = k \mid Z_{n-1} > 0]. \end{aligned} \quad (35)$$

By part (i) of Lemma 3.6, we can find some $\delta > 0$ and $k_0 \in \mathbb{N}$ such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$,

$$\mathbb{P}^{\mu_2} \left[\left| \frac{1}{k} \sum_{i=1}^k X_{n-1,i} - m_{\mu_2} \right| \geq \eta \right] \leq \varepsilon/2 \quad \text{for all } k \geq k_0. \quad (36)$$

From (35) and (36) we obtain that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$,

$$\begin{aligned} & \mathbb{P}^{\mu_2} [|\hat{m}_n - m_{\mu_2}| \geq \eta \mid Z_{n-1} > 0] \\ & \leq \varepsilon/2 + \sum_{k=1}^{k_0} \mathbb{P}^{\mu_2} \left[\left| \frac{1}{k} \sum_{i=1}^k X_{n-1,i} - m_{\mu_2} \right| \geq \eta \right] \mathbb{P}^{\mu_2} [Z_{n-1} = k \mid Z_{n-1} > 0] \\ & \leq \varepsilon/2 + \sum_{k=1}^{k_0} \mathbb{P}^{\mu_2} [Z_{n-1} = k \mid Z_{n-1} > 0]. \end{aligned} \quad (37)$$

By part (i) of Lemma 3.3 we can find some $n_0 \in \mathbb{N}$ (and decrease the $\delta > 0$ chosen above if necessary) such that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$ and $n \geq n_0$,

$$\mathbb{P}^{\mu_2}[Z_{n-1} = k | Z_{n-1} > 0] \leq \varepsilon / (2k_0) \quad \text{for all } k = 1, \dots, k_0. \quad (38)$$

Now, (37)–(38) yield that for all $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$ and all $n \geq n_0$,

$$\mathbb{P}^{\mu_2}[|\hat{m}_n - m_{\mu_2}| \geq \eta | Z_{n-1} > 0] \leq \varepsilon.$$

This implies (34).

Part (ii) can be shown analogously. Use parts (ii) instead of (i) of Lemmas 3.6 and Lemma 3.3, and remove the restriction $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$ everywhere. \square

5 Proof of Theorem 2.4

Note that (uniform) robustness of (\hat{m}_n) on $\mathcal{N} \subset \mathcal{N}_1^1$ in the sense of Definition 2.1 holds if and only if (\hat{m}_n) is both (uniformly) asymptotically and (uniformly) finite sample robust on \mathcal{N} in the following sense.

Definition 5.1 (i) *The sequence (\hat{m}_n) is said to be asymptotically robust on \mathcal{N} if for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \rho(\mathbb{P}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1}) \leq \varepsilon \quad \text{for all } n \geq n_0.$$

It is said to be uniformly asymptotically robust on \mathcal{N} if δ can be chosen independently of $\mu_1 \in \mathcal{N}$.

(ii) *The sequence (\hat{m}_n) is said to be finite sample robust on \mathcal{N} if for every $\mu_1 \in \mathcal{N}$, $n \in \mathbb{N}$, and $\varepsilon > 0$ there is some $\delta > 0$ such that*

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \rho(\mathbb{P}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1}) \leq \varepsilon. \quad (39)$$

It is said to be uniformly finite sample robust on \mathcal{N} if δ can be chosen independently of $\mu_1 \in \mathcal{N}$.

The claim of Theorem 2.4 is an immediate consequence of Theorems 5.3, 5.4, and 5.5 below. We first require the following lemma. Write $\mathbb{P}_A^\mu[\cdot] := \mathbb{P}^\mu[\cdot | A]$ for any $\mu \in \mathcal{N}_1^1$ and $A \in \mathcal{F}$.

Lemma 5.2 (i) *Let $\mathcal{N} \subset \mathcal{N}_1^1$ be any set such that $m_\mu > 1$ for all $\mu \in \mathcal{N}$. Then the sequence (\hat{m}_n) is asymptotically robust on \mathcal{N} if and only if for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\begin{aligned} \mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \\ \implies \quad \rho(\mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_2} \circ \hat{m}_n^{-1}) \leq \varepsilon \quad \text{for all } n \geq n_0. \end{aligned} \quad (40)$$

(ii) Let $\mathcal{N} \subset \mathcal{N}_1^1$ be a uniformly ψ_1 -integrating set with $\inf_{\mu \in \mathcal{N}} m_\mu > 1$. Then the sequence (\hat{m}_n) is uniformly asymptotically robust on \mathcal{N} if and only if for every $\varepsilon > 0$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \mu_1, \mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \\ \implies \rho(\mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_2} \circ \hat{m}_n^{-1}) \leq \varepsilon \quad \text{for all } n \geq n_0. \end{aligned} \quad (41)$$

Proof We start by proving part (i). First assume that (40) holds. By (40) and part (i) of Lemma 3.4 we obtain that for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$, and $A \in \mathcal{B}(\mathbb{R}_+)$,

$$\begin{aligned} \mathbb{P}^{\mu_1} \circ \hat{m}_n^{-1}[A] &= \mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_1} \circ \hat{m}_n^{-1}[A] \cdot \mathbb{P}^{\mu_1}[Z_{n-1} > 0] + \delta_0[A] \cdot \mathbb{P}^{\mu_1}[Z_{n-1} = 0] \\ &\leq (\mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_2} \circ \hat{m}_n^{-1}[A^\varepsilon] + \varepsilon) \cdot (\mathbb{P}^{\mu_2}[Z_{n-1} > 0] + \varepsilon) + \delta_0[A^\varepsilon] \cdot (\mathbb{P}^{\mu_2}[Z_{n-1} = 0] + \varepsilon) \\ &\leq \mathbb{P}_{\{Z_{n-1} > 0\}}^{\mu_2} \circ \hat{m}_n^{-1}[A^\varepsilon] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} > 0] + \delta_0[A^\varepsilon] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} = 0] + 3\varepsilon + \varepsilon^2 \\ &= \mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1}[A^\varepsilon] + 3\varepsilon + \varepsilon^2 \\ &\leq \mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1}[A^{(3\varepsilon + \varepsilon^2)}] + (3\varepsilon + \varepsilon^2). \end{aligned}$$

Hence, we can find for every $\mu_1 \in \mathcal{N}$ and $\tilde{\varepsilon} > 0$ some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \implies \rho(\mathbb{P}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1}) \leq \tilde{\varepsilon} \quad \text{for all } n \geq n_0.$$

This means that (\hat{m}_n) is asymptotically robust on \mathcal{N} .

Now assume that the sequence (\hat{m}_n) is asymptotically robust on \mathcal{N} . It suffices to show that (40) holds when the Prohorov metric ρ is replaced by the bounded Lipschitz metric

$$\beta(\mu_1, \mu_2) := \sup_{h \in \text{BL}_1} \left| \int h d\mu_1 - \int h d\mu_2 \right|,$$

where BL_1 is the set of all functions $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\|h\|_{\text{BL}} := \|h\|_{\text{L}} + \|h\|_{\infty} \leq 1$ with $\|h\|_{\text{L}} := \sup_{x \neq y} |h(x) - h(y)|/|x - y|$ and $\|h\|_{\infty} := \sup_x |h(x)|$; following the instructions on p.398 in [8] it can be easily shown that $\rho^2 \leq \frac{3}{2}\beta$. By the asymptotic robustness and part (i) of Lemma 3.4 we obtain that for every $\mu_1 \in \mathcal{N}$ and $\varepsilon \in (0, (1 - q_{\mu_1})/2)$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $\mu_2 \in \mathcal{N}$ with

$$d_{\text{TV}}(\mu_1, \mu_2) \leq \delta,$$

$$\begin{aligned}
& \beta(\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} \circ \hat{m}_n^{-1}) \\
& \leq \sup_{h \in \text{BL}_1} \left| \frac{\int h d\mathbb{P}^{\mu_1} \circ \hat{m}_n^{-1}}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0]} - \frac{\int h d\mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1}}{\mathbb{P}^{\mu_2}[Z_{n-1} > 0]} \right| + \left| \frac{\mathbb{P}^{\mu_1}[Z_{n-1} = 0]}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0]} - \frac{\mathbb{P}^{\mu_2}[Z_{n-1} = 0]}{\mathbb{P}^{\mu_2}[Z_{n-1} > 0]} \right| \\
& \leq \frac{\sup_{h \in \text{BL}_1} \left| \int h d\mathbb{P}^{\mu_1} \circ \hat{m}_n^{-1} - \int h d\mathbb{P}^{\mu_2} \circ \hat{m}_n^{-1} \right|}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} > 0]} + \frac{|\mathbb{P}^{\mu_2}[Z_{n-1} > 0] - \mathbb{P}^{\mu_1}[Z_{n-1} > 0]|}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} > 0]} \\
& \quad + \frac{|\mathbb{P}^{\mu_1}[Z_{n-1} = 0] - \mathbb{P}^{\mu_2}[Z_{n-1} = 0]|}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} > 0]} + \frac{|\mathbb{P}^{\mu_2}[Z_{n-1} > 0] - \mathbb{P}^{\mu_1}[Z_{n-1} > 0]|}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} > 0]} \\
& \leq 4 \frac{\varepsilon}{\mathbb{P}^{\mu_1}[Z_{n-1} > 0] \cdot \mathbb{P}^{\mu_2}[Z_{n-1} > 0]} \leq 4 \frac{\varepsilon}{(1 - q_{\mu_1}) \cdot (1 - q_{\mu_1} - \varepsilon)}. \tag{42}
\end{aligned}$$

This implies that (40) holds (for the bounded Lipschitz metric).

Part (ii) can be shown analogously. Use part (ii) instead of (i) of Lemma 3.4. Replace the last bound in (42) by $4 \frac{\varepsilon}{(1-q_{\mu_1}) \cdot (1-q_{\mu_2})}$, which is less than or equal to $4\varepsilon/p^2$ by part (ii) of Lemma 3.2 (further decreasing $\delta > 0$ if necessary). \square

Theorem 5.3 (i) *The sequence (\hat{m}_n) is asymptotically robust on any locally uniformly ψ_1 -integrating set $\mathcal{N} \subset \mathcal{N}_1^1$ with $m_\mu > 1$ for all $\mu \in \mathcal{N}$.*

(ii) *The sequence (\hat{m}_n) is uniformly asymptotically robust on any uniformly ψ_1 -integrating set $\mathcal{N} \subset \mathcal{N}_1^1$ with $\inf_{\mu \in \mathcal{N}} m_\mu > 1$.*

Proof We first prove part (i). By Lemma 5.2(i) it suffices to show that for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
& \mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \\
& \implies \rho(\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} \circ \hat{m}_n^{-1}) \leq \varepsilon \quad \text{for all } n \geq n_0. \tag{43}
\end{aligned}$$

Fix $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$. For every μ_2 we have

$$\begin{aligned}
& \rho(\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_1} \circ \hat{m}_n^{-1}, \mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} \circ \hat{m}_n^{-1}) \\
& \leq \rho(\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_1} \circ \hat{m}_n^{-1}, \delta_{m_{\mu_1}}) + |m_{\mu_1} - m_{\mu_2}| + \rho(\delta_{m_{\mu_2}}, \mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} \circ \hat{m}_n^{-1}). \tag{44}
\end{aligned}$$

We start with the first and third summands in this bound. By part (i) of Theorem 4.1 we can find some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$,

$$\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} [|\hat{m}_n - m_{\mu_2}| \leq \varepsilon/3] > 1 - \varepsilon/3.$$

Since $\{\hat{m}_n \in A\} \subset \{m_{\mu_2} \in A^{\varepsilon/3}\} \cup \{|\hat{m}_n - m_{\mu_2}| > \varepsilon/3\}$ for every $A \in \mathcal{B}(\mathbb{R}_+)$, we obtain for every $A \in \mathcal{B}(\mathbb{R}_+)$ that

$$\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} \circ \hat{m}_n^{-1}[A] \leq \delta_{m_{\mu_2}}[A^{\varepsilon/3}] + \varepsilon/3,$$

and hence

$$\rho(\mathbb{P}^{\mu_2}_{\{Z_{n-1} > 0\}} \circ \widehat{m}_n^{-1}, \delta_{m_{\mu_2}}) \leq \varepsilon/3.$$

For the second summand on the right-hand side of (44) we use the fact that $\mu \mapsto m_\mu$ is $(d_{\text{TV}}, |\cdot|)$ -continuous at μ_1 , shown in Lemma 3.1(i). Decreasing $\delta > 0$ above further if necessary, we obtain

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad |m_{\mu_1} - m_{\mu_2}| \leq \varepsilon/3. \quad (45)$$

This completes the proof of part (i).

Part (ii) can be shown analogously. Use parts (ii) instead of (i) of Lemma 5.2, Theorem 4.1 and Lemma 3.1. Note that a finite $\delta > 0$ is only needed for the analogue of (45) (not before). \square

Theorem 5.4 (i) *The sequence (\widehat{m}_n) is finite sample robust on $\mathcal{N} := \mathcal{N}_1^1$.*

(ii) *The sequence (\widehat{m}_n) is uniformly finite sample robust on any uniformly ψ_1 -integrating set $\mathcal{N} \subset \mathcal{N}_1^1$.*

Proof We start by proving part (i). We have to show that for every $\mu_1 \in \mathcal{N}$, $\varepsilon > 0$, and $n \in \mathbb{N}$ there is some $\delta > 0$ such that

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad \rho(\mathbb{P}^{\mu_1} \circ \widehat{m}_n^{-1}, \mathbb{P}^{\mu_2} \circ \widehat{m}_n^{-1}) \leq \varepsilon. \quad (46)$$

By the simple direction in Strassen's theorem (e.g. Theorem 2.13 in [15]) the right-hand side in (46) holds if we can find a probability measure $\nu = \nu_{\mu_1, \mu_2}$ on $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ such that

$$\nu \circ \pi_i^{-1} = \mathbb{P}^{\mu_i} \circ \widehat{m}_n^{-1}, \quad i = 1, 2, \quad (47)$$

(where $\pi_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is the projection on the i th coordinate) and

$$\nu[\{(x_1, x_2) \in \mathbb{R}_+^2 : |x_1 - x_2| \leq \varepsilon\}] \geq 1 - \varepsilon. \quad (48)$$

Thus, for part (i) it suffices to show that for every $\mu_1 \in \mathcal{N}$, $\varepsilon > 0$, and $n \in \mathbb{N}$ there is some $\delta > 0$ such that for every $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$ one can find a probability measure ν on $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ satisfying (47)–(48).

Let $\mu_1 \in \mathcal{N}$, $\varepsilon > 0$, and $n \in \mathbb{N}$ be fixed. By Lemma 3.5 we can find some $\delta > 0$ such that

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad d_{\text{TV}}^{(2)}(\mathbb{P}^{\mu_1} \circ (Z_{n-1}, Z_n)^{-1}, \mathbb{P}^{\mu_2} \circ (Z_{n-1}, Z_n)^{-1}) \leq \varepsilon.$$

Together with Strassen's theorem this implies that for every $\mu_2 \in \mathcal{N}$ with $d_{\text{TV}}(\mu_1, \mu_2) \leq \delta$ there is some probability measure $\widetilde{\nu}$ on $(\mathbb{N}_0^2 \times \mathbb{N}_0^2, \mathfrak{P}(\mathbb{N}_0^2 \times \mathbb{N}_0^2))$ such that

$$\widetilde{\nu} \circ \widetilde{\pi}_i^{-1} = \mathbb{P}^{\mu_i} \circ (Z_{n-1}, Z_n)^{-1}, \quad i = 1, 2, \quad (49)$$

(where $\tilde{\pi}_i : \mathbb{N}_0^2 \times \mathbb{N}_0^2 \rightarrow \mathbb{N}_0^2$ is the projection on the i th coordinate) and

$$\tilde{\nu} \left[\left\{ (z_{n-1}^1, z_n^1, z_{n-1}^2, z_n^2) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2 : \|(z_{n-1}^1, z_n^1) - (z_{n-1}^2, z_n^2)\| \leq \varepsilon \right\} \right] \geq 1 - \varepsilon, \quad (50)$$

where $\|\cdot\|$ denotes the standard Euclidean norm. Now, we set $\hat{m}_n^*(Z_{n-1}, Z_n) := Z_n/Z_{n-1}$ such that $\hat{m}_n = \hat{m}_n^*(Z_{n-1}, Z_n)$, define

$$\nu := \tilde{\nu} \circ (\hat{m}_n^* \circ \tilde{\pi}_1, \hat{m}_n^* \circ \tilde{\pi}_2)^{-1}. \quad (51)$$

From (49) we obtain for $i = 1, 2$

$$\begin{aligned} \nu \circ \pi_i^{-1} &= (\tilde{\nu} \circ (\hat{m}_n^* \circ \tilde{\pi}_1, \hat{m}_n^* \circ \tilde{\pi}_2)^{-1}) \circ \pi_i^{-1} \\ &= \tilde{\nu} \circ (\pi_i \circ (\hat{m}_n^* \circ \tilde{\pi}_1, \hat{m}_n^* \circ \tilde{\pi}_2))^{-1} \\ &= \tilde{\nu} \circ (\hat{m}_n^* \circ \tilde{\pi}_i)^{-1} \\ &= (\tilde{\nu} \circ \tilde{\pi}_i^{-1}) \circ \hat{m}_n^{*-1} \\ &= (\mathbb{P}^{\mu_i} \circ (Z_{n-1}, Z_n)^{-1}) \circ \hat{m}_n^{*-1} \\ &= \mathbb{P}^{\mu_i} \circ (\hat{m}_n^* \circ (Z_{n-1}, Z_n))^{-1} \\ &= \mathbb{P}^{\mu_i} \circ \hat{m}_n^{-1}. \end{aligned}$$

That is, (47) holds for ν defined in (51). Further, if $\|(z_{n-1}^1, z_n^1) - (z_{n-1}^2, z_n^2)\| < 1$, then $(z_{n-1}^1, z_n^1) = (z_{n-1}^2, z_n^2)$ and so $\hat{m}_n(z_{n-1}^1, z_n^1) = \hat{m}_n(z_{n-1}^2, z_n^2)$. Thus, assuming without loss of generality $0 < \varepsilon < 1$, we obtain

$$\begin{aligned} &\nu \left[\{(x_1, x_2) \in \mathbb{R}_+^2 : |x_1 - x_2| > \varepsilon\} \right] \\ &= \tilde{\nu} \left[\{(z_{n-1}^1, z_n^1, z_{n-1}^2, z_n^2) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2 : |\hat{m}_n^*(z_{n-1}^1, z_n^1) - \hat{m}_n^*(z_{n-1}^2, z_n^2)| > \varepsilon\} \right] \\ &\leq \tilde{\nu} \left[\{(z_{n-1}^1, z_n^1, z_{n-1}^2, z_n^2) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2 : (z_{n-1}^1, z_n^1) \neq (z_{n-1}^2, z_n^2)\} \right] \\ &< \varepsilon, \end{aligned}$$

where the last step is ensured by (50). That is, we also have (48) for ν defined in (51). This completes the proof of part (i).

Part (ii) can be shown analogously. Take into account that, under the stronger assumption on \mathcal{N} , Lemma 3.5 and part (ii) of Lemma 3.1 imply that the mapping $\mathcal{N}_1^1 \rightarrow (\mathcal{N}_1^1)^2$, $\mu \mapsto \mathbb{P}^\mu \circ (Z_{n-1}, Z_n)^{-1}$ is *uniformly* $(d_{TV}, d_{TV}^{(2)})$ -continuous. \square

Theorem 5.5 *Let $\mathcal{N} \subset \mathcal{N}_1^1$ such that $m_\mu > 1$ for all $\mu \in \mathcal{N}$, and assume that there exists some $\mu_1 \in \mathcal{N}$ such that the mapping $\mathcal{N} \ni \mu \mapsto m_\mu$ is not $(d_{TV}, |\cdot|)$ -continuous at μ_1 . Then the sequence (\hat{m}_n) is not asymptotically robust on \mathcal{N} .*

Proof Suppose that the sequence (\widehat{m}_n) is asymptotically robust on \mathcal{N} . In view of the identity $\min\{1; |m_{\mu_1} - m_{\mu_2}|\} = \rho(\delta_{m_{\mu_1}}, \delta_{m_{\mu_2}})$, we have for every $\mu_2 \in \mathcal{N}$,

$$\begin{aligned} & \min\{1; |m_{\mu_1} - m_{\mu_2}|\} \\ & \leq \rho(\mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_1} \circ \widehat{m}_n^{-1}, \mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_2} \circ \widehat{m}_n^{-1}) + \sum_{i=1}^2 \rho(\delta_{m_{\mu_i}}, \mathbb{P}_{\{Z_{n-1}>0\}}^{\mu_i} \circ \widehat{m}_n^{-1}) \\ & =: S_0(n, \mu_1, \mu_2) + \sum_{i=1}^2 S_i(n, \mu_i). \end{aligned}$$

Let $\varepsilon > 0$ be fixed. Recall that ρ metrizes the weak topology. Thus, using Theorem 4.1(ii) with $\mathcal{N} = \{\mu_i\}$, we can find some $n_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^2 S_i(n, \mu_i) \leq \varepsilon/2 \quad \text{for all } n \geq n_1.$$

By the asymptotic robustness of (\widehat{m}_n) and part (i) of Lemma 5.2, we can also find some $\delta > 0$ and $n_0 \geq n_1$ such that

$$\mu_2 \in \mathcal{N}, \quad d_{\text{TV}}(\mu_1, \mu_2) \leq \delta \quad \implies \quad S_0(n, \mu_1, \mu_2) \leq \varepsilon/2 \quad \text{for all } n \geq n_0.$$

Thus, the mapping $\mu \mapsto m_\mu$ is $(d_{\text{TV}}, |\cdot|)$ -continuous at μ_1 . This contradicts the assumption. \square

6 Extension to general initial states

In this section, we outline modifications in the arguments that show that our main result, Theorem 2.4, is true when we start the process with a population of general size z_0 . Note that in this case, we can decompose the process (Z_n) into z_0 independent processes $(Z_n^{(i)})$ started with 1 individual for $i = 1, \dots, z_0$ such that

$$Z_n = Z_n^{(1)} + \dots + Z_n^{(z_0)}. \quad (52)$$

In order to avoid confusion we write $\mathbb{P}^{z_0, \mu}$ for the probability measure under which (Z_n) started in z_0 with offspring distribution μ evolves. Denoting by $q_\mu^{(z_0)}$ the extinction probability of (Z_n) , it is immediate that $q_\mu^{(z_0)} = q_\mu^{z_0} \leq q_\mu$ for all $z_0 \in \mathbb{N}$. We will show that Theorems 5.3 and 5.4 hold also for (Z_n) started in a general z_0 such that Theorem 2.4 follows.

Theorem 5.3 uses Lemma 5.2 whose proof works in the same way as before: We simply have to note that Lemma 3.4 still holds due to the inequality

$$\begin{aligned} |\mathbb{P}^{z_0, \mu_1}[Z_n = 0] - \mathbb{P}^{z_0, \mu_2}[Z_n = 0]| &= |\mathbb{P}^{\mu_1}[Z_n = 0]^{z_0} - \mathbb{P}^{\mu_2}[Z_n = 0]^{z_0}| \\ &\leq z_0 |\mathbb{P}^{\mu_1}[Z_n = 0] - \mathbb{P}^{\mu_2}[Z_n = 0]| \end{aligned}$$

and replace q_{μ_i} by $q_{\mu_i}^{z_0}$ for $i = 1, 2$ in the argument.

The other result that is needed in Theorem 5.3 is Theorem 4.1. The proof of the latter still applies as long as Lemma 3.3 holds. The modifications here are the following: According to (15), replacing q_{μ_2} by $q_{\mu_2}^{z_0}$ we need to bound $\mathbb{P}^{z_0, \mu_2}[Z_n = k|A]$ and $\mathbb{P}^{z_0, \mu_2}[B|Z_n > 0]$ where A is the event of survival and B that of extinction of (Z_n) . For the former we use that for all $z_0 \in \mathbb{N}$,

$$\mathbb{P}^{z_0, \mu_2}[Z_n = k|A] \leq \mathbb{P}^{z_0, \mu_2}[Z_n \leq k|A] \leq \mathbb{P}^{\mu_2}[Z_n \leq k|A]$$

in (18). Replacing again q_{μ_2} by $q_{\mu_2}^{z_0}$ we see by Bayes Formula that for the latter it suffices to consider

$$\mathbb{P}^{z_0, \mu_2}[Z_n > 0|B] \leq \sum_{i=1}^{z_0} \mathbb{P}^{z_0, \mu_2}[Z_n^{(i)} > 0|B] \leq z_0 \mathbb{P}^{\mu_2}[Z_n^{(i)} > 0|B_i],$$

where B_i denotes the event of extinction of $Z_n^{(i)}$. The last probability is bounded appropriately in the proof of Lemma 3.3.

Having established the validity of Theorem 5.3 we turn to Theorem 5.4. Here, the essential ingredient is the analogous version of Lemma 3.5. However, it is easy to see that (26) holds with $C_n(\mu_1, \mu_2)$ replaced by $z_0 C_n(\mu_1, \mu_2)$: Namely, note that due to (52)

$$\begin{aligned} & d_{\text{TV}}^{(n)}(\mathbb{P}^{z_0, \mu_1} \circ (Z_1, \dots, Z_n)^{-1}, \mathbb{P}^{z_0, \mu_2} \circ (Z_1, \dots, Z_n)^{-1}) \\ &= \frac{1}{2} \sum (\mathbb{P}^{z_0, \mu_1}[Z_1^{(1)} = z_1^{(1)}, \dots, Z_n^{(1)} = z_n^{(1)}, \dots, Z_1^{(z_0)} = z_1^{(z_0)}, \dots, Z_n^{(z_0)} = z_n^{(z_0)}] \\ & \quad - \mathbb{P}^{z_0, \mu_2}[Z_1^{(1)} = z_1^{(1)}, \dots, Z_n^{(1)} = z_n^{(1)}, \dots, Z_1^{(z_0)} = z_1^{(z_0)}, \dots, Z_n^{(z_0)} = z_n^{(z_0)}]), \end{aligned}$$

where the sum ranges over all $z_1^{(1)}, \dots, z_n^{(1)}, \dots, z_1^{(z_0)}, \dots, z_n^{(z_0)} \in \mathbb{N}_0$. But due to the independence of $(Z_n^{(1)})$ to $(Z_n^{(z_0)})$ we have

$$\begin{aligned} & \mathbb{P}^{z_0, \mu_1}[Z_1^{(1)} = z_1^{(1)}, \dots, Z_n^{(1)} = z_n^{(1)}, \dots, Z_1^{(z_0)} = z_1^{(z_0)}, \dots, Z_n^{(z_0)} = z_n^{(z_0)}] \\ &= \prod_{i=1}^{z_0} \mathbb{P}^{\mu_1}[Z_1^{(i)} = z_1^{(i)}, \dots, Z_n^{(i)} = z_n^{(i)}] \end{aligned}$$

so that it follows with (28) that

$$\begin{aligned} & d_{\text{TV}}^{(n)}(\mathbb{P}^{z_0, \mu_1} \circ (Z_1, \dots, Z_n)^{-1}, \mathbb{P}^{z_0, \mu_2} \circ (Z_1, \dots, Z_n)^{-1}) \\ & \leq z_0 d_{\text{TV}}^{(n)}(\mathbb{P}^{\mu_1} \circ (Z_1, \dots, Z_n)^{-1}, \mathbb{P}^{\mu_2} \circ (Z_1, \dots, Z_n)^{-1}). \end{aligned}$$

The conclusion follows now with the original version of Lemma 3.5. This completes the proof of Theorem 5.4 and thus also of Theorem 2.4.

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